Exercise 34

Solve the Stokes problem which is concerned with the unsteady boundary layer flows induced in a semi-infinite viscous fluid bounded by an infinite horizontal disk at $z = 0$ due to nontorsional oscillations of the disk in its own plane with a given frequency ω. The equation of motion and the boundary and initial conditions are

$$\begin{aligned} u_t &= \nu u_{zz}, \quad z > 0, \ t > 0, \\ u(z,t) &= Ue^{i\omega t} \quad \text{on } z = 0, \ t > 0, \\ u(z,t) &\to 0 \quad \text{as } z \to \infty \text{ for } t > 0, \\ u(z,0) &= 0 \quad \text{for } t \leq 0 \text{ and } z > 0, \end{aligned}$$

where $u(z,t)$ is the velocity of the fluid of kinematic viscosity ν and U is constant. Solve the Rayleigh problem ($\omega = 0$). Explain the physical significance of both the Stokes and Rayleigh solutions. [TYPO: This should be t.]

Solution

Solution to the Stokes Problem

The PDE is defined for $t > 0$ and we have an initial condition, so the Laplace transform can be used to solve it. It is defined as

$$\mathcal{L}\{u(z,t)\} = \bar{u}(z,s) = \int_0^t e^{-st} u(z,t)\, dt,$$

which means the derivatives of u with respect to z and t transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial^n u}{\partial z^n}\right\} &= \frac{d^n \bar{u}}{dz^n} \\ \mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} &= s\bar{u}(z,s) - u(z,0) \end{aligned}$$

Take the Laplace transform of both sides of the PDE.

$$\mathcal{L}\{u_t\} = \mathcal{L}\{\nu u_{zz}\}$$

The Laplace transform is a linear operator.

$$\mathcal{L}\{u_t\} = \nu \mathcal{L}\{u_{zz}\}$$

Transform the derivatives with the relations above.

$$s\bar{u}(z,s) - u(z,0) = \nu \frac{d^2\bar{u}}{dz^2}$$

From the initial condition, $u(z,t) = 0$ for $t \leq 0$, we have $u(z,0) = 0$.

$$\frac{d^2\bar{u}}{dz^2} = \frac{s}{\nu}\bar{u}(z,s)$$

The PDE has thus been reduced to an ODE whose solution can be written in terms of exponential functions.

$$\bar{u}(z,s) = A(s)e^{\sqrt{\frac{s}{\nu}}z} + B(s)e^{-\sqrt{\frac{s}{\nu}}z}$$

In order to satisfy the condition that $u(z,t) \to 0$ as $z \to \infty$, we require that $A(s) = 0$.

$$\bar{u}(z,s) = B(s)e^{-\sqrt{\frac{s}{\nu}}z}$$

To determine $B(s)$ we have to use the boundary condition at $z = 0$, $u(0,t) = Ue^{i\omega t}$. Take the Laplace transform of both sides of it.

$$\mathcal{L}\{u(0,t)\} = \mathcal{L}\{Ue^{i\omega t}\}$$

$$\bar{u}(0,s) = \frac{U}{s - i\omega} \tag{1}$$

Setting $z = 0$ in the formula for $\bar{u}$ and using equation (1), we have

$$\bar{u}(0,s) = B(s) = \frac{U}{s - i\omega}.$$

Thus,

$$\bar{u}(z,s) = \frac{U}{s - i\omega}e^{-\sqrt{\frac{s}{\nu}}z}.$$

Now that we have $\bar{u}(z,s)$, we can get $u(z,t)$ by taking the inverse Laplace transform of it. The convolution theorem can be used to write an integral solution for $u(z,t)$. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

The inverse Laplace transform of the individual functions are

$$\mathcal{L}^{-1}\left\{\frac{U}{s - i\omega}\right\} = Ue^{i\omega t}$$

$$\mathcal{L}^{-1}\left\{e^{-\sqrt{\frac{s}{\nu}}z}\right\} = \frac{z}{\sqrt{4\pi\nu t^3}}e^{-\frac{z^2}{4\nu t}},$$

so by the convolution theorem, we have for $u(z,t)$

$$u(z,t) = \mathcal{L}^{-1}\{\bar{u}(z,s)\} = \int_0^t Ue^{i\omega(t-\tau)}\frac{z}{\sqrt{4\pi\nu\tau^3}}e^{-\frac{z^2}{4\nu\tau}}\,d\tau.$$

Bring the constants out in front and write the integral like so.

$$u(z,t) = \frac{Uze^{i\omega t}}{\sqrt{4\pi\nu}}\int_0^t \frac{1}{\tau^{3/2}}e^{-\frac{z^2}{4\nu\tau} - i\omega\tau}\,d\tau \tag{2}$$

Evaluating the integral and simplifying, we get

$$u(z,t) = \frac{Ue^{i\omega t}}{2}\left[e^{-\sqrt{\frac{i\omega}{\nu}}z}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}} - \sqrt{i\omega t}\right) + e^{\sqrt{\frac{i\omega}{\nu}}z}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}} + \sqrt{i\omega t}\right)\right],$$

where erfc is the complementary error function, a known special function, defined as

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-r^2}\,dr. \tag{3}$$

In order to satisfy the condition that $u(z,t)=0$ for $t\leq 0$, we write the solution as a piecewise function.

$$u(z,t)=\begin{cases}0 & t\leq 0\\ \frac{Ue^{i\omega t}}{2}\left[e^{-\sqrt{\frac{i\omega}{\nu}}z}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}-\sqrt{i\omega t}\right)+e^{\sqrt{\frac{i\omega}{\nu}}z}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}+\sqrt{i\omega t}\right)\right] & t>0\end{cases}$$

This can be written compactly with the Heaviside function. Therefore,

$$u(z,t)=\frac{Ue^{i\omega t}}{2}\left[e^{-\sqrt{\frac{i\omega}{\nu}}z}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}-\sqrt{i\omega t}\right)+e^{\sqrt{\frac{i\omega}{\nu}}z}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}+\sqrt{i\omega t}\right)\right]H(t).$$

Solution to the Rayleigh Problem

In the event $\omega=0$, equation (2) becomes

$$u(z,t)=\frac{Uz}{2\sqrt{\pi\nu}}\int_0^t\frac{1}{\tau^{3/2}}e^{-\frac{z^2}{4\nu\tau}}\,d\tau.$$

Make the substitution,

$$p=\frac{z}{\sqrt{4\nu\tau}}$$

$$dp=-\frac{z}{4\sqrt{\nu\tau^3}}\,d\tau\quad\rightarrow\quad -\frac{4\sqrt{\nu}}{z}\,dp=\frac{1}{\tau^{3/2}}\,d\tau.$$

The solution becomes

$$u(z,t)=\frac{Uz}{2\sqrt{\pi\nu}}\int_\infty^{\frac{z}{\sqrt{4\nu t}}}e^{-p^2}\left(-\frac{4\sqrt{\nu}}{z}\,dp\right).$$

Bring the constants out in front of the integral and use the minus sign to switch the limits of integration.

$$u(z,t)=\frac{2U}{\sqrt{\pi}}\int_{\frac{z}{\sqrt{4\nu t}}}^\infty e^{-p^2}\,dp$$

Using equation (3), we can write this in terms of erfc.

$$u(z,t)=U\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right)$$

To satisfy the last condition, $u(z,t)=0$ for $t\leq 0$, we include the Heaviside function. Therefore, when $\omega=0$,

$$u(z,t)=U\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right)H(t).$$

We could have also just set $\omega=0$ in the solution to the Stokes problem to get this result.

The answer at the back of the book for the Stokes problem is totally off. Also, the solution for the Rayleigh problem doesn't have $H(t)$, which means it is only valid for $t>0$.